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LOGIC 2.0

A *derealistic* refundation of logic.

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1 — LOGIC 1.0

- **Rests upon Trinity *Semantics/Syntax/Meta*.**

  Meta: sort of *go-between* linking *reality* and *language*.

  Ensures that reality is *faithfully* described.

- **Seems convincing; indeed *deceiptive*.**

  Kizhe variables: clerical mistake, a variable not used for *generalisation*.

  Yields logical blunder $\forall \Rightarrow \exists$.

  « Fixed » by declaring empty models « fake news ».

- **Logic 1.0 is a sort of *axiomatic realism*.**

  Axiomatic means *military*, not quite rational.

  Logic based upon *distrust* of misleading « reality ».

- **Logic 2.0 replaces trinity with *knitting*.**

	EXPLICIT	IMPLICIT
ANALYTIC	$\mathbf{1 - \mathit{Constat}}$	$\mathbf{2 - \mathit{Performance}}$
SYNTHETIC	$\mathbf{3 - \mathit{Usine}}$	$\mathbf{4 - \mathit{Usage}}$

I — Proof-nets: from 1.0 to 2.0

2 — Original problem

- « Natural deduction » for linear logic.

  Linear negation makes tree-shaped proofs *obsolete*.

  Hypothesis written as *conclusion*.

  Several conclusions: problem of *sequentialisation*.

- Solved for *multiplicative* fragment $\otimes, \parr, \sim$.

  Links: Axiom, Cut, Times, Par: (0,2), (2,0), (2,1), (2,1).

  Switches: position L/R for Par-links.

  Correctness: connected/acyclic *(tree)* for any position of switches.

- *Sequentialisation theorem:* reduction of *correct* nets to sequent calculus.

- Difficult to extend to *full logic*.

  Boxes used in 1986 version to handle additives. . .

  Commutative conversions: a pain in the ass.

  Jump criterions: depend on the proof-net, no *duality*.

- 1.0 misconception: proofnets seen as a syntactic *convenience*.

3 — FLUNKED JAILBREAKS

- **Multiplicative proofs and tests as *permutation* of atoms.**

  Passing a test: σ passes τ iff $\sigma\tau$ *cyclic*.

  Orthogonality: $\sigma \perp \tau \ := \sigma\tau$ cyclic.

  Negation: becomes orthogonality, $\sim A := A^{\perp}$.

- ***Geometry of interaction*** **(GoI, 1988) uses operator (vN) algebras.**

  Permutations replaced with *partial symmetries*: $\sigma = \sigma^3 = \sigma^*$.

  Orthogonality: various notions, e.g., $\sigma\tau$ nilpotent.

- ***Ludics*** **(2000) based upon additives and *focalisation*.**

- **Both approaches « hegelian » : contradictory foundations.**

  $\sim A$ tests A (and conversely).

  Semantic (alethic) *refutation* replaced with (deontic) *recusation*.

  Gospel: the judges will be judged.

- **Ends in a mess: one can *never* be sure of anything!**

  BHK aporia: how do we know that a proof is actually a proof?

4 — CONDITIONS OF POSSIBILITY

- **How do I *know* that a proof is a proof?**

  Typical case: « Axiom » link, i.e., $\vdash \sim A, A$.

  GoI subpoenas *all* proofs of $\vdash A$ and $\vdash \sim A$.

  Hegelian duality must be fixed by *finite* preorthogonal.

- **L'*usine* (= factory), the missing piece of logic 1.0.**

  Proof-nets: the typical occurrence of *usine*.

  Herbrand's theorem: early prefiguration of usine (1930).

- **Analogy: *disk* vs. *player*.**

  Test of disk (resp. player) by means of *testing* player (resp. record).

  Test of testing record by testing player succeeds.

  Justifies $\vdash \mathrm{disk}, \mathrm{player}$.

- **Complementarity of testings need not extend to *tested*.**

  Testing devices zone-free: *tested* player may refuse *tested* disk.

  Cut between $\vdash \Gamma, \mathrm{disk}$ and $\vdash \mathrm{player}, \Delta$ may fail.

II — MULTIPLICATIVES

5 — Proofs as partitions

- **Two candidates for multiplicative *analytics:***

  Flows (directed): $A \rightsquigarrow B$, **from** A **go to** B.

  Identity link as $A \rightsquigarrow \sim A \; + \; \sim A \rightsquigarrow A$.

  Graphs (undirected): « edge » $\{A, B\}$ **between** A **and** B.

  Identity link as $\{A, \sim A\}$.

- **Original version (flows) leads to *permutations* of *literals:***

  No short trip condition translates as:

  Duality proofs/switchings: $\sigma \perp \tau$ **iff** $\sigma\tau$ ***cyclic*.**

  Unitary operators eventually generalise permutations: GoI.

- **Danos-Regnier: duality through *bipartite* graphs *proof/switching*.**

  Links $B = \{b_1, \ldots, b_k\} / C = \{c_1, \ldots, c_l\}$ **as *vertices* of graph.**

  $\mathrm{card}(\{b_1, \ldots, b_k\} \cap \{c_1, \ldots, c_l\}) \leq 1$.

  Literals $\alpha_1, \ldots, \alpha_n$ **as *edges* of bipartite graph.**

  Edge α **between** B **and** C **iff** $B \cap C = \{\alpha\}$.

  Correctness: bipartite graph *connected* and *acyclic*.

6 — The preorthogonal

- Literals $\alpha_1, \ldots, \alpha_n$ of A replaced with *support* $|A| := \{1, \ldots, n\}$.

  Proof of A: *red* partition σ of $|A|$.

  Switching of A: *cyan* partition σ of $|A|$.

  Negation: corresponds to exchange between *red* and *cyan*.

- $\sigma \in A$ iff $\sigma \perp \tau$ (i.e., $\sigma \cup \tau$ *connected* and *acyclic*) for all $\tau \in \sim A$.

  Conversely: $\tau \in \sim A$ iff $\sigma \perp \tau$ for all $\sigma \in A$.

- *Preorthogonal* $A^p \subset \sim A$ with « enough » tests: l'*usine*.

  From $\tau \in A^p, \upsilon \in B^p$ form $\tau \cup \upsilon \in (A \parr B)^p$.

  From $T \in \tau \in A^p, U \in \upsilon \in B^p$ form

  $(\tau \setminus \{T\}) \cup (\upsilon \setminus \{U\}) \cup \{T \cup U\}) \in (A \otimes B)^p$.

  Multiplicative neutrals $1, \perp$, a 1.0 contraption: $n \neq 0$.

- Identity « axiom » : if $\tau \in A^p, \upsilon \in (\sim A)^p$, then $\tau \perp \upsilon$ *(usine)*.

- Cut rule: if $\sigma \perp A^p$ and $\rho \perp (\sim A)^p$, then $\sigma \perp \rho$ *(usage)*.

  Proves *cut-elimination:* knitting *usine/usage*.

7 — NORMALISATION

- **Lewis Carroll's *flunked* cut-elimination (1893):**

$$\frac{\vdash \Gamma, A \quad \vdash \sim A, \Delta}{\vdash \Gamma, \Delta} \quad \text{replaced with} \quad \frac{\vdash \Gamma, A \quad \vdash \sim A, \Delta}{\vdash \Gamma, A \otimes \sim A, \Delta}$$

  New cut with $A \multimap A$: Achilles *flees* from Tortoise!

  Reduces cut-elimination to case $\Gamma = \emptyset$ *(Modus Ponens)*.

- **Function $\sigma \in A \multimap B$ applied to *argument* $\rho \in A$ yields $\sigma(\rho) \in B$.**

  Change colour: $\rho \in A$ replaced with $\rho \in A$.

  Contract edges in $\rho \cup \sigma$: if $\{2, 5\} \in \rho$, let S_2, S_5 s.t. $\{i\} \cup S_i \in \sigma$;

  Replace $\{2, 5\} + \{2\} \cup S_2 + \{5\} \cup S_5$ with $S_2 \cup S_5$.

  Identity: $\iota := \{\{i, n+i\}; 1 \leq i \leq n\} \in A \multimap A$; $\iota(\rho) = n + \rho$.

- ***L'usine*** **(orthogonality to $A^p, (A \multimap B)^p$) guarantees *l'usage:***

  No deadlock: S_2, S_5 disjoint (acyclicity).

  No vanishing: $S_2 \cup S_5 \neq \emptyset$ (connectedness).

  Logical correctness: $\sigma(\rho) \in B$.

III — Truth

8 — HEGELIAN NEGATION

- **1.0 negation is *alethic*, concerns *truth*.**
  Negation as *refutation* within *format* proceeding from the Sky.
  Consistency: formula and negation not *both* provable.
- **2.0 negation is *deontic*, concerns the *format* itself.**
  Negation as *recusation:* « objection overruled ».
  Hegel's *contradictory foundations:* inconsistent according to 1.0 logic.
  Everything provable, at least as a switching of negation.
- **Need to *revisit* the notion of *truth*.**
  Tarski: $A \wedge B$ true when A true *and* B true, etc.
  Amounts at: A true when A true.
- **Distinguish, among the proofs of A, between:**
  Ordeals: general proofs of sole *deontic* value, possible tests for $\sim A$.
  True proofs: among ordeals, those of *alethic* value, who convey certainty.
- **Truth (of proofs) preserved by the full *usine:* logical rules and *cut*.**
  Consistency: some formula, e.g., 0, without true proof.

9 — Truth as binarity

- Usual logical proofs begin with identity « axioms » $\vdash A, \sim A$.

  Binarity condition: partition π true when made of cells of size 2.

- Binarity preserved by cut-elimination: if $\{2,5\} \in \rho$, $\{i\} \cup S_i \in \sigma$, then $S_i = \{s_i\}$ and $S_2 \cup S_5 = \{s_2, s_5\}$.

- Binarity ensures *consistency:*

  If $\sigma \perp \tau$ and σ binary, then τ not binary.

- Notion not suitable for *second order:*

  Logical proof of $\exists X A$ contains *subjective* witness T s.t. $A[T/X]$.

  Witness is a correctness condition, no reason to be binary.

- Split support $|A|$ as a disjoint union $|A|_o + |A|_s$.

  Cell $S \in \sigma$ *objective* if $s \subset |A|_o$, *subjective* if $S \subset |A|_s$.

  Non animist partition: all cells either objective or subjective.

  Truth of σ : *non animist* and *objective component* $\sigma \upharpoonright |A|_o$ binary.

- *Non animist binarity* suitable for usual logic.

10 — The topological (sub)invariant, a.k.a. gain

- ***Euler-Poincaré*** **invariant of a graph** G.

  $\sharp G := \mathrm{card}(\mathrm{vertices}) - \mathrm{card}(\mathrm{edges})$.

  Theorem: $\sharp G = \mathrm{card}(\mathrm{components}) - \mathrm{card}(\mathrm{cycles})$.

  Tree: **connected and acyclic, hence** $\sharp G = 1$.

- **Logical duality: define** $\sharp\sigma$ **and** $\sharp\tau$ **s.t.** $2\cdot\sharp(\sigma \cup \tau) = \sharp\sigma + \sharp\tau$.

  $\sharp\sigma := 2\cdot\mathrm{card}\,\sigma - \mathrm{card}\,|A|$, $\sharp\tau := 2\cdot\mathrm{card}\,\tau - \mathrm{card}\,|A|$.

  Orthogonality: **if** $\sigma \perp \tau$, **then** $\sharp\sigma + \sharp\tau = 2$.

  $\sharp\sigma = \sum_{s\in\sigma} \sharp S$, **with** $\sharp\{s_1, \ldots, s_k\} := 2 - k$.

- **Extend invariant to** ***subinvariant,*** **the** ***gain,*** **taking care of subjectivity.**

  Objective cell: $\sharp\{s_1, \ldots, s_k\} := 2 - k$; ***subjective cell:*** $\sharp S := 0$.

  Non animist binary **partition** σ : $\sharp\sigma = 0$.

  Animist cell: $\sharp(S_o + S_s) := \sharp S_o - 2$,

  i.e, $-k$ **where** k **is the number of objective elements of** S.

- **If** $\sigma \perp \tau$, **then** $\sharp\sigma + \sharp\tau \leq 2$: ***gain*** **may** ***increase*** **during normalisation.**

  Truth: σ ***true*** **iff** $\sharp\sigma \geq 0$. **Normalisation** ***reinforces*** **truth.**

11 — フ AND ヲ

- **The real constants of logic: *atomic* (one point) propositions.**

  Objective フ **or *subjective*** ヲ.

  Both self-dual and true. Unique partition $\{\{\alpha\}\}$ **receives value:**

  $\sharp\text{フ} := 1$, $\qquad \sharp\text{ヲ} := 0$.

  Proof-net $\{\text{フ}, \text{ヲ}\}$ **logically correct, but *false* (value** -1**).**

- **Multiplicative combinations of the sole** フ **:**

  Up to equivalence, one combination フ_n **s.t.** $\sharp\text{フ}_n = n$.

  $\text{フ}_1 := \text{フ}$; **for** $n > 0$, $\text{フ}_{n+1} := \text{フ}_n \otimes \text{フ}$.

  For $n \leq 1$, $\text{フ}_{n-1} := \text{フ}_n \parr \text{フ}$, **e.g.,** $\text{フ}_0 := \text{フ} \parr \text{フ}$.

- **Multiplicative combinations of** $\text{フ}, \text{ヲ}$ **with at least one** ヲ **:**

  Up to equivalence, one combination $\hat{n}$ **s.t.** $\sharp\hat{n} = n$.

  $\text{フ}_n \otimes \text{ヲ} \equiv \hat{n} \equiv \text{フ}_{n+2} \parr \text{ヲ}$.

  The series フ_n **and** $\hat{n}$ **distinct.**

  Only relation: $\text{フ}_n \multimap \hat{n} \multimap \text{フ}_{n+2}$.

- ***Partitions* definitely better than *permutations*.**

12 — Basic Presburger arithmetic

- **Multiplicative behaviour of the $\urcorner_n$:**

$$\urcorner_m \otimes \urcorner_n \equiv \urcorner_{m+n}, \qquad \urcorner_m \parr \urcorner_n \equiv \urcorner_{m+n-2}.$$
$$\sim\urcorner_n \equiv \urcorner_{2-n}, \qquad \urcorner_m \multimap \urcorner_n \equiv \urcorner_{n-m}.$$

- **Multiplicative behaviour of the $\hat{n}$:**

$$\hat{m} \otimes \hat{n} \equiv \hat{m} \parr \hat{n} \equiv \widehat{n+m}.$$
$$\sim\hat{n} \equiv \widehat{-n}, \qquad \hat{m} \multimap \hat{n} \equiv \widehat{n-m}.$$

- **Mixed multiplicative behaviour:**

$$\widehat{m} \otimes \urcorner_n \equiv \widehat{m+n}, \qquad \widehat{m} \parr \urcorner_n \equiv \widehat{m+n-2}.$$
$$\widehat{m} \multimap \urcorner_n \equiv \widehat{n-m-2}, \qquad \urcorner_m \multimap \hat{n} \equiv \widehat{n-m}.$$

- **Absurdity 0 defined as $!\widehat{-1} \otimes \Finv$, i.e., $\sim(\widehat{-1} \Rightarrow \hat{0})$.**

  Falsity A ***false*** **when $\neg A$ (i.e., $A \Rightarrow 0$)** ***true*****.**

  Truth: $\hat{n}$ true for $n \geq 0$, $\neg\hat{n}$ true for $n < 0$.

  Order: defined by $m \multimap n$; ***true*** **when $m \leq n$,** ***false*** **when $n < m$.**

- **However, product $m \cdot n$ makes no sense in terms of the $\urcorner_n$ and $\hat{n}$.**

13 — A JAILBREAK

- **Jailbreak from tarskism and the idea of *subliminal classicism*.**
  Constructivity sort of *guilding the lily* over classical frame.
- **Good news: topological truth *refutes* classical logic.**
  Excluded middle: $\hat{m} \equiv \hat{n} \vee \hat{n} \equiv \hat{p} \vee \hat{p} \equiv \hat{m}$.
  Contradicted by: $\neg(\hat{m} \equiv \hat{n})$ **for** $m \neq n$.
- **Deviation w.r.t. *classical truth:***

A	B	$A \otimes B$	$A ⅋ B$	$\sim A$
$\mathbf{t}$	$\mathbf{t}$		$\mathbf{f}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	

Disjunction more deviant: linear negation does not exchange true/false.

- **Jailbreak from the very idea of *truth tables*.**
  $\urcorner_n$ **and** $\hat{n}$ **receive same value** n.
  Inequivalent: $\hat{n} \multimap \urcorner_n \equiv \widehat{-2}$, **false.**

14 — Digression: games

- Games in logic: Gentzen (unpublished) « consistency proof » (1936).

  Propositions as games.

  Proofs as winning strategies.

- Mistreated as 1.0 « game semantics » (Lorenzen, Lorenz, Felscher, etc.)

  Rule proceeding from the Sky.

  Status of *Opponent* dubious.

  Ad hoc: sort of carbon copy of syntax.

- Ludics, etc. consider sort of *deontic* game.

  Player, opponent free to interact, provided play *converges*.

  Opponent may play *losing* for the sole sake of *forbidding* move of *Player*.

- Present in proof-nets: *deontic* interaction $\sigma \perp \tau$. Three notions of *gain:*

  Play: $\sharp_\tau(\sigma)$. Does not depend upon τ in multiplicative case.

  Strategy: $\sharp_A\sigma := \inf_{\tau\in\sim A} \sharp_\tau(\sigma)$. May take value $-\infty$.

  Game: $\sharp A := \sup_{\sigma\in A} \sharp_A\sigma$. May take values $-\infty, +\infty$.

IV — The four horsemen of cognition

15 — A KANTIAN TWIST

	Explicit	Implicit
Analytic	Constat	Performance
Synthetic	Usine	Usage

16 — ANALYTICS

- **Central role of l'*usine*, i.e., proof-nets.**

  Location $p_A(x)$, ***sublocation*** $p_A(\mathtt{l}\cdot x)$ **: where propositions belong.**

  Delogicalised: A **and** $\sim A$ **same slot (untyped).**

  Star: sort of « thick wire » between n ***rays*** $(n = 1, 2, 3, \ldots)$.

  Splits into substars, subsubstars, using variables, the *same* for all rays.

  Constellation: finite combination $\sum \lambda_i \mathcal{S}_i$, **with** $\lambda_i > 0$ **real numbers.**

- ***Dynamics*** **should be internal: *self-performing,* down with the *meta!***

  Plugging: use of *complementary* colours, e.g., red/cyan, green/magenta.

  Matching: the analytics of cut-elimination, a.k.a. *normalisation*.

  $\lambda[\![\,\Gamma, t\,]\!] + \mu[\![\,u, \Delta\,]\!] \rightsquigarrow \lambda\mu[\![\,\Gamma\theta, \Delta\theta\,]\!]$, **with** θ **m.g.m. of** t, u.

- **Normalisation of *constellations* as *colour-elimination*.**

  Church-Rosser: equivalence between *one and two pairs* of colours.

  Major knitting responsible for the *associativity* of logical operations.

  Constat: uncolored constellations *(normal, explicit)*.

  Performance: coloured constellations *(colour-elimination, implicit)*.

17 — SYNTHETICS

- **Type, format, *logic*. Distinction explicit/implicit, i.e., *a posteriori, a priori*.**

  A posteriori: passing of finite battery of tests. *Usine*, cut-free.

  Non analytic: only in the very *choice* of tests.

  A priori: plugging with unknown complementary artifact. *Usage*, cut rule.

  Synthetic implicit refers to the monstrosity of *all* possible uses.

- **L'usine should guarantee l'usage, *modulo* a « cut-elimination » result.**

  Sequentialisation: no longer central; exotic *non sequential* connectives.

  Adequation: the *tested* are complementary, i.e., testing is *sharp* enough.

  Hilbert's consistency: miscarriage of kantism, no *checking* of the *a priori!*

  Apodictic cheques (absolutely safe): mere *impossibility*.

- ***Knitting* usine/usage very demanding. We thus discover that:**

  Church-Rosser permutates cuts (associativity).

  Switches must be *local,* i.e., independent of each other.

  And independent from the proof-net tested (no « jump » criterion).

  Analytics: *finite sets* $\rightsquigarrow$ *linear combinations* (ensures additive knitting).

18 — THE CRITERION

- Propositions $A, B, C, \ldots$ *located* as $p_A(x), p_B(x), p_C(x), \ldots$

  Proof σ in red tested by test τ in cyan and uncoloured (conclusion).

  Test succeeds when $\sigma + \tau$ admits (uncoloured) normal form $p_\Gamma(x) := [\![\, p_A(x); x \in \Gamma \,]\!]$.

  Variants $p_\Gamma(t)$, etc. excluded because of *socialisation* (tensorisation).

- *Weakening* (absence) and *contraction* (repetition) would induce *variants*.

  Neutral $\perp$ impossible; alternative second order $\perp := \exists X (X \otimes {\sim} X)$.

  Exponentials as logical *ions* (like $\mathrm{OH}^-, \mathrm{NH}^{4+}$).

  Combined in $!A \otimes B$ and $?A \parr B$, $A \Rightarrow B$.

  Pure exponentials available at second order: $\forall X((A \Rightarrow X) \multimap X)$.

  Hidden conclusions: $\Gamma, \underline{\Delta}$. Result still $p_\Gamma(x) := [\![\, p_A(x); x \in \Gamma \,]\!]$.

- $?A \parr B$ handled like $\parr$ without left position of switch.

  Compensate absent position with *modest* switching, devoted to *acyclicity*.

  Modest test may use modest positions; result either $\emptyset$ or $p_\Gamma(x)$.

  Connect $?A \otimes B$ with $?A$; *ignore* (erase) B.

19 — ATOMS フ, ヲ VS. VARIABLES

- **Propositional atoms $P, Q, R, \ldots$ and negations $\sim P, \sim Q, \sim R, \ldots$**

  1.0 blunder: $P, Q, R, \ldots$ **as « constants ».**

  Variables $X, Y, Z, \ldots$ **universally quantified.**

  Quantifiers $\forall X, \forall Y, \forall Z, \ldots$ **in *implicit* prefix.**

- **Links restricted to $\{X, \sim X\}$: $\{X, X\}$, $\{X, Y\}$, etc. forbidden.**

  1.0 approach: treat them like like *axioms* proceeding from the Sky.

  2.0 approach: use switchings of quantifiers.

- **Switching of $\forall X$: involves three positions.**

  1: $X :=$ フ ⅋ フ **and** $\sim X :=$ フ $\otimes$ フ.

  2: $X :=$ フ $\otimes$ フ **and** $\sim X :=$ フ ⅋ フ.

  3: $X :=$ フ **and** $\sim X :=$ フ.

- **Positions $1, 2$ forbid $\{X, X\}$, $\{X, Y\}$, etc.**

- **Position 3 forces connection between « full » X and $\sim X$.**

  Otherwise normal form would no longer be the full $p_\Gamma(x)$.

20 — ETA-EXPANSION

- *η-conversion,* a marginal rewriting rule: $\lambda x \cdot t(x) \rightsquigarrow t$.
  Surjective pairing: $(\pi_1 t, \pi_2 t) \rightsquigarrow t$, etc.
  Academic use: add *tedious* and *straightforward* section in shallow paper.
  Inspiration: 0%, *transpiration:* 100%!
- Better handled reversed: *eta-expansion,* $t \rightsquigarrow \lambda x \cdot t(x)$.
  Complies with category-theoretic *doxa* (universal problems).
  Poor analytics: only a *rewriting,* not self-performing.
- *Proof nets:* η as *decomposition* of non-atomic *identities*.
  Replace $[\![A \parr B, \sim A \otimes \sim B]\!]$ with $[\![A, \sim A]\!] + [\![B, \sim B]\!]$.
  Switching assumes everything η-expanded.
  Works in non-expanded case.
  Testing performs its *own* η-expansion.
- Typical knitting: the test τ does not depend upon σ.
  Duality: $\sigma \perp \tau$ would not make sense otherwise.

V — ADDITIVES

21 — ADDITIVE NEUTRALS

- **The *weakest* point of linear logic original.**

  1.0 version insists upon seeing $\top$ as *final* element of category.

  Wavering methodology: diverging constraints, nothing definite.

- **Second order definitions $\top := \exists X\, X, \quad 0 := \forall X\, X$.**

  Unilateral: don't use both of $X, \sim X$.

  Balance rights/duties $X/\sim X$ not at stake.

  However presence of *subjective* elements.

- **Boils down to $\top := (\text{フ} \parr \text{ヲ}) \Rightarrow \text{ヲ}, \quad 0 := !(\text{フ} \parr \text{ヲ}) \otimes \text{ヲ}$.**

  Extremal gains: $\sharp 0 = -\infty, \quad \sharp\top = +\infty$.

- $\dfrac{\vdash \Gamma, A}{\vdash \Gamma, \top}$ **relocation of part A of proof-net σ, including switching τ of A.**

  σ in フ.

  τ (upper part of switching) in left ヲ.

  τ (lower part of same) in right ヲ.

22 — INTUITIONISTIC DISJUNCTION

- **Logical rules (*introductions* and *elimination*):**

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}$$

Second order: **witness *extraneous* C in elimination.**

$$A \vee B := \forall X((A \Rightarrow X) \Rightarrow ((B \Rightarrow X) \Rightarrow X)).$$

- **Standard normalisation *(introduction/elimination):***

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{A \vee B} \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C} \\ \vdots \end{array} \quad \rightsquigarrow \quad \begin{array}{c}\vdots\\ A\\ \vdots\\ C\\ \vdots\end{array}$$

23 — Commutative conversions

- ***Subformula*** **property fails.**

  Extraneous $C \Rightarrow D$ **may** ***hide*** **cut.**

  Lewis Carroll: premise $\forall X(X \Rightarrow X)$ **(and variants) as cut.**

- ***Commutative*** **conversions:**

  Eliminations below $\vee$**-elim. commute « above », e.g., :** $\vee / \Rightarrow$**:**

$$\dfrac{\begin{matrix}\vdots\\ C\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\quad \begin{matrix}[A]\\ \vdots\\ C\Rightarrow D\end{matrix}\quad \begin{matrix}[B]\\ \vdots\\ C\Rightarrow D\end{matrix}}{C\Rightarrow D}}{D} \quad\rightsquigarrow\quad \dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\qquad \dfrac{\begin{matrix}\vdots\\ C\end{matrix}\quad \begin{matrix}[A]\\ \vdots\\ C\Rightarrow D\end{matrix}}{D}\qquad \dfrac{\begin{matrix}\vdots\\ C\end{matrix}\quad \begin{matrix}[B]\\ \vdots\\ C\Rightarrow D\end{matrix}}{D}}{D}$$

  Church-Rosser: not that bad, but a pain in the ass!

  Rewriting: not self-executing, i.e., not ***analytic*****.**

  Knitting: poor, must be refused.

24 — ADDITIVE PROOF-NETS

- **Basic problem:** ***superposition*** **of** ***contexts*** Γ **in** $\&$ **rule.**

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B}$$

Analogue of the two auxiliary premises of $\vee$-elimination $(C \rightsquigarrow \Gamma)$.
Locative conflict: both Γ want to occupy same slot.
Boxes: mimick $\vee$-elimination; lead to complex *commutative conversions*.
Boolean weights: *left* Γ vs. *right* Γ : poorly knitted.

- ***Coherent*** **analytics (coherence between** ***stars*****).**

Superposition handled by incoherent copies of Γ.
Correctness by means of a spectacular criterion.
Analytics a bit unmanageable; globally a pain in the ass.

- **Main problem,** ***methodology:*** **too many constraints.**

Inherited from 1.0 tradition. Some may be obsolete.
Sequentialisation: replaced with *cut-elimination*.
Subformula property: must be *relaxed*.

25 — ADDITIVE CONJUNCTION

- **Based on *analytic substrate* of second order version:**

  $A \,\&\, B := \exists X(!(X \multimap A) \otimes !(X \multimap B) \otimes X)$.

  $A \oplus B := \forall X((A \multimap X) \Rightarrow ((B \multimap X) \Rightarrow X))$.

- **Five sublocations** $\Phi_{\mathtt{L}}, \Phi_{\mathtt{R}}, \Phi_{\mathtt{l}}, \Phi_{\mathtt{r}}, \Phi_{\mathtt{m}}$ **of** $\Phi = A \,\&\, B, A \oplus B$ **:**

  $q_\Phi(\mathtt{L} \cdot x), q_\Phi(\mathtt{R} \cdot x)$ **: correspond to subformulas** A, B.

  $p_\Phi(\mathtt{l} \cdot x), p_\Phi(\mathtt{r} \cdot x), p_\Phi(\mathtt{m} \cdot x)$ **: correspond to the three** X.

- ***Analytisation* (delogicalisation) of context** Γ **: if** $C \in \Gamma$,

  $p_C(x) \rightsquigarrow p_\Phi(\mathtt{l} \cdot (\mathtt{c} \cdot x)), p_\Phi(\mathtt{r} \cdot (\mathtt{c} \cdot x)), p_\Phi(\mathtt{m} \cdot (\mathtt{c} \cdot x))$.

  Three copies of Γ **devoid of logical significance, i.e., *unswitched*.**

  Premise $\vdash \Gamma, A$ **(resp.** $\vdash \Gamma, B$**) of** $\&$ **rule in** $\Phi_{\mathtt{l}}, \Phi_{\mathtt{L}}$ **(resp.** $\Phi_{\mathtt{r}}, \Phi_{\mathtt{R}}$**).**

  Third component: identity link (delocation) between Γ **and** $\Phi_{\mathtt{m}}$.

- ***Plain switching L/R* of** $\Phi = A \,\&\, B$**, e.g., *left:***

  Connect conclusion Φ **with premise** $A = \Phi_{\mathtt{L}}$**; and** $\Phi_{\mathtt{l}}$ **with** $\Phi_{\mathtt{m}}$ **:**

  $[\![\, p_\Phi(x), q_\Phi(\mathtt{L} \cdot x) \,]\!] + [\![\, p_\Phi(\mathtt{l} \cdot x), p_\Phi(\mathtt{m} \cdot x) \,]\!]$.

  Modest switching (left): $[\![\, p_\Phi(x), p_\Phi(\mathtt{l} \cdot x), p_\Phi(\mathtt{m} \cdot x) \,]\!]$.

26 — ADDITIVE DISJUNCTION

- $$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$$

  Left rule interpreted by locating A as $\Phi_{\mathtt{L}}$ together with:

  Identity link between $\Phi_{\mathtt{l}}, \Phi_{\mathtt{m}}$: $[\![p_\Phi(\mathtt{l}\cdot x), p_\Phi(\mathtt{m}\cdot x)]\!]$.

- *Plain* switching $\oplus_\lambda$ (first use of > 0 weights ; $\lambda = 1, 2$ are enough).

  $\lambda[\![p_\Phi(x), p_\Phi(\mathtt{m}\cdot x)]\!]+$
  $\lambda^{-1}[\![q_\Phi(\mathtt{L}\cdot x), p_\Phi(\mathtt{l}\cdot x)]\!] + \lambda^{-1}[\![q_\Phi(\mathtt{R}\cdot x), p_\Phi(\mathtt{r}\cdot x)]\!]$.

- Six *modest* switchings $\oplus_{\mathtt{Ll}}, \oplus_{\mathtt{Lm}}, \oplus_{\mathtt{L}}, \oplus_{\mathtt{Rl}}, \oplus_{\mathtt{Rm}}, \oplus_{\mathtt{R}}$ e.g., :

  $\oplus_{\mathtt{Ll}}$: $[\![p_\Phi(x), q_\Phi(\mathtt{L}\cdot x), p_\Phi(\mathtt{l}\cdot x)]\!] + [\![p_\Phi(\mathtt{m}\cdot x)]\!]$.
  $\oplus_{\mathtt{Lm}}$: $[\![p_\Phi(x), q_\Phi(\mathtt{L}\cdot x), p_\Phi(\mathtt{m}\cdot x)]\!] + [\![p_\Phi(\mathtt{l}\cdot x)]\!]$.
  $\oplus_{\mathtt{L}}$: $[\![p_\Phi(x), q_\Phi(\mathtt{L}\cdot \mathtt{x})]\!]$.

- If τ_λ uses $\oplus_\lambda(\lambda = 1, 2)$, $\sigma + \tau_\lambda$ reduces to *left* or *right* form, e.g., :

  Left form: $[\![p_\Gamma(x), q_\Phi(\mathtt{L}\cdot \mathtt{x})]\!] + [\![p_\Phi(\mathtt{l}\cdot x), p_\Phi(\mathtt{m}\cdot x)]\!]+$
  $\lambda[\![p_\Phi(x), p_\Phi(\mathtt{m}\cdot x)]\!] + \lambda^{-1}[\![q_\Phi(\mathtt{L}\cdot x), p_\Phi(\mathtt{l}\cdot x)]\!]$.

27 — NORMALISATION

- **Cut between $\Phi = A \,\&\, B$ and $\sim\Phi$: bracketed conclusion $[\Phi \otimes \sim\Phi]$.**

$$\dfrac{\dfrac{A \quad \mathtt{l} \quad \mathtt{m} \quad \mathtt{r} \quad B}{\Phi} \qquad \dfrac{\sim A \quad \sim\mathtt{l} \quad \sim\mathtt{m} \quad \sim\mathtt{r} \quad \sim B}{\sim\Phi}}{[\Phi \otimes \sim\Phi]}$$

Locations $\mathtt{l}, \mathtt{r}, \mathtt{m}$ **split into finitely many *similar* sublocations** $\mathtt{l}_i, \mathtt{m}_i \ (i \in I)$ **and** $\mathtt{r}_j, \mathtt{m}_j \ (j \in J)$ **let** $K := I \cup J, P := I \cap J$

Context splits into $\ldots, \Gamma_p, \ldots, \Delta$ **with** $\Gamma_p = \emptyset$ **for** $p \notin K$**.**

Proof σ **splits into *components:***

$\&$: $\{A, \ldots, \mathtt{l}_i, \ldots\}, \{B, \ldots, \mathtt{r}_j, \ldots\}, \ldots, \{\Gamma_p, \mathtt{m}_p\}, \ldots$

$\oplus$: **either** $\{\sim A, \Delta\}, \{\sim\mathtt{l}, \sim\mathtt{m}\}$ **or** $\{\sim B, \Delta\}, \{\sim\mathtt{r}, \sim\mathtt{m}\}$**.**

- **Plugging of $\mathtt{l}, \sim\mathtt{l}$ and $\mathtt{r}, \sim\mathtt{r}$ and $\mathtt{m}, \sim\mathtt{m}$:**

Puts together either: $\{A, \ldots, \Gamma_k, \ldots\}$ ***cut with*** $\{\sim A, \Delta\}$**.**

Or: $\{B, \ldots, \Gamma_k, \ldots\}$ ***cut with*** $\{\sim B, \Delta\}$**.**

- **Cut on $A \,\&\, B$ replaced with cut on A (or cut on B).**

28 — THE SUBFORMULA PROPERTY

- **Important, although slightly *ad hoc* from the very beginning:**

  Predicate calculus: $A[t/x]$ **subformula of** $\forall x A$**.**

  Controls formulas appearing in cut-free proof.

  Second order: definite loss of subformula property, i.e., of *any* control.

- **Our additives do enjoy *subformula property*, provided we define:**

  $A, B, \mathtt{l}, \mathtt{r}, \mathtt{m}$ **as subformulas of** $A \,\&\, B, A \oplus B$**.**

- **However** $A \,\&\, B$ **may « hide » cut:**

  Premise σ **of** $\vdash A, [C \otimes \sim C]$ **located in** $\mathtt{L}, \mathtt{l}_i$**.**

  Identity $C \multimap C$ **located in** $\mathtt{m}_i$**.**

  Left switch connects $C \otimes \sim C$ **with** $C \multimap C$ **by *performing* the cut.**

- **Cut not quite hidden, since *implicitly* eliminated by correctness.**

  Real second order can hide a « bad » (non normalising) cut.

  Analyticised version does *perform* the cut, *slicewise:* no bad surprise!

- **Best *knitting* for additives: *simpler* than coherent version.**

29 — SEQUENTIALISATION

- **No longer part of the main *knitting*.**
  Replaced with adequation *usine/usage,* a.k.a. normalisation.
  Prejudice: everything should be written *step by step*.
  Useful (very), but by no means *essential*.
- n**-ary multiplicative: set of partitions of** $\{1, \ldots, n\}$.
  Duality: $\mathcal{C} \perp \mathcal{D}$ **iff union is a *tree*.**
  Multiplicative: non-trivial set of partitions equal to *bidual*.
  Example: $\otimes := \{\{1, 2\}\}$ **vs.** $⅋ := \{\{1\}, \{2\}\}$.
- ***Sequentialisable* connectives: built from** $\otimes, ⅋$ ***(series/parallel)*.**
  Exotic 4-ary $¶ := \{\{1, 2\}, \{3, 4\}\} + \{\{2, 3\}, \{4, 1\}\}$.
  Orthogonal: $\sim ¶ := \{\{1, 3\}, \{2\}, \{4\}\} + \{\{2, 4\}, \{1\}, \{3\}\}$.
  Non sequential: $¶, \sim ¶$ **admit proof-nets, but *no sequent calculus*.**
- ***Open question:* are non sequential connectives important?**
  Didn't yet succeed in finding a positive use for them.
  Hard to handle, hence prognosis « reserved ».

30 — Full propositional calculus

- Consists of multiplicatives, additives and ***exponentials***.
  Devoted to ***weakening*** (absence) and ***contraction*** (repetition).
- Three exponentials s.t. $!A \multimap \Downarrow A \multimap \downarrow A$.
  Plain (strong) exponentials $!$, $?$ allow ***weakening*** and full ***contraction***.
  Auxiliary variable takes care of copies.
  Expansionals $\Downarrow, \Uparrow$ allows weakening and limited contraction.
  Same as above, but no auxiliary variable; enough for ***neutral additives***.
  Affine version $\downarrow, \uparrow$ only allows weakening.
  Enough for second order definition of ***additives***.
- Duplication of tests: duplicated switches must stay ***independent***.
  Fixed by means of ***non uniform*** modest switchings.
- Problem with $\downarrow(A \otimes B) \multimap A$:
  Fixed by weighted $⅋$, e.g., $\lambda[\![\, q_{A ⅋ B}(x), q_A(x) \,]\!] + \lambda^{-1}[\![\, q_B(x) \,]\!]$.
- ***Desaxiomatisation*** of arithmetic: third and fourth Peano axioms fixed.
  Recurrence: still a bit axiomatic, i.e., ***ad hoc.***