KEIO, 28 Novembre 2015

TRANSCENDENTAL SYNTAX

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The reconciliation of the *mathematical* and *philosophical* sides of logic thanks to *informatics*.

1 — EPICYCLES, A.K.A. THE REALIST PREJUDICE

- XX$^{\mathrm{th}}$ century logic begins *after* incompleteness.

  Herbrand: synthetic *a posteriori*, a.k.a. *usine*.
  BHK: synthetic *a priori*, a.k.a. *usage*.
  Gentzen: relation usine/usage through *cut-elimination*.

- XIX$^{\mathrm{th}}$ century, up to ~ 1925: axomatic and semantic.

  Hilbert: *militarism* (axiomatics). *A priori* $\rightsquigarrow$ consistency.
  Russell: *religion* (of reality). Semantics, a.k.a. *prejudice*.

- *Realism:* cognitive simplicism, yields monsters.

  Epicycles: fantasmatic reality backing *geocentric* prejudice.

- Realism expressed by *classical* reduction to *true/false*.

  Loss of propositional expressivity.
  Compensation: fantasmatic first-order individuals.
  Symptom: no logical handling of *equality* (next talk).

	Analytic	Synthetic
Explicit	Constat	Usine
Implicit	Performance	Usage

I — WHAT IS AN ANSWER?

Keywords: analytic, untyped, computational.

2 — ANALYTICITY : CONSTAT VS. PERFORMANCE

- Cognition *without* presupposition: everything on the table.

  Including table: finite (no etc.), no link to external « *reality* ».
  Verbatim: the style of cowards, *meaningless*.

- Key ↲ either *constative:* adds new line, *incremental*. Or:

  Performative: launches program, *destructive*.

- *Pure* lambda-calculus approximates analyticity.

  Strong normalisation relates constat and performance.
  Undecidability: performance $\neq$ constat ; no *pravdameter*.
  Church-Rosser relates performance and usage.

- *External* performance replaced with *self-performance:*

  Plugging of wires of complementary colours.
  Unification: wires split into *implicit* subwires.
  Resolution: clause $\Gamma \vdash A$ becomes $\{\gamma, a\}$.

3 — STARS AND CONSTELLATIONS

- ***Star:*** $n \neq 0$ **terms *(rays)* with exactly the same variables.**

  Disjoint: rays pairwise not *matchable*.

  Substitution: $[\![t_1, \ldots, t_n]\!]\theta := [\![t_1\theta, \ldots, t_n\theta]\!]$ **still a star.**

- ***Constellation:*** **finite set of stars.**

  Bound variables, i.e., local to each star.

  Rays of the (stars of the) constellation pairwise disjoint.

- ***Colours:*** **just a convenience, unary function letters.**

  Disjoint: come by *complementary* pairs.

  Pairs: `green`/`magenta`, `red`/`cyan`, `blue`/`yellow`.

- **Colours responsible for divide *constat/performance*.**

  Constative constellation: in black (no colour).

  Performance: elimination of colour, normalisation.

  GoI: analytic substrate of synthetic *cut-elimination*.

4 — STRONG NORMALISATION

- ***Diagrams*** **of constellation:** ***tree*** **(connected/acyclic graph).**

  Vertices: **stars (with repetitions). Infinitely many diagrams.**

  Edges: **formal equalities** $t = u, t = u, t = u$.

- ***Actualisation*** **of a diagram:**

  Match **underlying terms:** $t = u$ **becomes** $t\theta = u\theta$.

  Failure **of most actualisations; diagram** ***correct*** **otherwise.**

- ***Strong normalisation:*** **knitting constat/performance.**

  1–Finiteness: **all diagrams of size** N, **hence** $\geq N$ **fail.**

  Excludes $[\![x, x]\!]$. ***Undecidability:*** **no way to predict** N.

  2–Openness: **no** ***closed*** **correct diagram (with no** ***free*** **ray).**

- ***Residual*** **star of correct diagram: its actualised** ***free*** **rays.**

  Normal form: **constellation of** ***uncoloured*** **residual stars.**

  Church-Rosser: **two pairs of complementary colours.**

5 — NON-DETERMINISM

- *Non-determinism* in constellations allows matching rays.
  Resolution: stars $\Gamma \vdash A$ or $\Gamma \vdash A$: a fine mess (PROLOG).
  π*-calculi* : parallel λ-calculus or cheap linear logic?
- *Matching rays* can only represent *Alzheimer*, NL-style.
  Coherence: $\mathcal{S} \ddagger \mathcal{T}$: *forbidden* substitutions.
  Strong normalisation: *self-incoherent* diagrams fail.
- $A \,\&\, B$: choose between « parallel universes » A/B.
  If already in universe A, I cannot see alternative B.
- Herbrand: *formal* function $f(t)$, a variable unknown to t.
  Herbrand boolean $\eta_{\mathcal{S}}$ indexed by a substar of some $\mathcal{T}$.
  Evolution of $\mathcal{T}$ into $\mathcal{T}'$ induces similar evolution $\eta_{\mathcal{S}} \rightsquigarrow \eta_{\mathcal{S}'}$.
- $\eta_{\mathcal{A}\&\mathcal{B}}$: boolean living « outside » A/B. Chooses A.
  Cancellation with $\neg\eta_{\mathcal{A}\&\mathcal{B}}$: only if behave in *same* way.
  Arrival in $A \,\&\, B$: not influenced by dichotimy A/B.

II — WHAT IS A QUESTION?

Keywords: synthetic, typed, logical.

6 — SYNTHETICITY : USINE VS. USAGE

- Cognition *with* presupposition. Dubious *since* meaningful.
- *L'usine* a.k.a. synthetic *a posteriori:* factory tests.

  Proof-nets: no vicious circle (already in Herbrand).

  Testing: analytic performance; output unquestionable.
- *L'usage*, a.k.a. synthetic *a priori:* use of the product.

  Gentzen: the cut-rule, deductive *since* destructive.
- Fundamental *duality* of meaning: *dinaturals*, hexagons.

  Predictivity: *commitment* usine w.r.t. usage.

  Cut-elimination: performance implementing the reduction.

  Incompleteness: convergence of reduction problematic.
- *Consistency proofs:* no commitment. *Ditto* with *realism:*

  Semantics: identification usine/usage: no testing.

  Reformed BHK: one must choose between testing and use.

7 — MULTIPLICATIVE PROOF-NETS

- **Function symbols** $\mathtt{l}, \mathtt{r}, \mathtt{g}$ **(0-ary),** $\cdot$ **binary.**

  To each proposition A **associate** ***location*** $p_A(x)$.

  To each proof π **associate** ***vehicle*** $\pi^\bullet$.

  Identity axiom $\vdash A, \sim A$: $\pi^\bullet := [\![p_A(x), p_{\sim A}(x)]\!]$.

- $p_A(x) := p_{A \maltese B}(\mathtt{l} \cdot x)$, $p_B(x) := p_{A \maltese B}(\mathtt{r} \cdot x)$ ($\maltese = \otimes, \parr, \ldots$)

  $\parr$**-rule: if** π **comes from** ν **of** $\vdash \Gamma, A, B$, $\pi^\bullet := \nu^\bullet$.

  $\otimes$**-rule: if** π **from** ν, μ **of** $\vdash \Gamma, A$, $\vdash \Delta, B$, **then** $\pi^\bullet := \nu^\bullet + \mu^\bullet$.

- ***Ordeals:*** $q_A(x) := p_A(\mathtt{g} \cdot x)$; **the** $q_A(x)$ **pairwise** ***disjoint***.

  Conclusions: $\mathtt{green}/\mathtt{black}$, **premises:** $\mathtt{magenta}/\mathtt{yellow}$.

- **LEGO** ***bricks:*** **Literals:** $[\![\frac{p_A(x)}{q_A(x)}]\!]$; **conclusion** $A \in \Gamma$: $[\![\frac{q_A(x)}{p_A(x)}]\!]$.

  $\otimes$**-link:** $[\![\frac{q_A(x), q_B(x)}{q_{A \otimes B}(x)}]\!]$.

  $\parr$**-links:** $[\![\frac{q_A(x)}{q_{A \parr B}(x)}]\!] + [\![\frac{q_B(x)}{}]\!]$ **or** $[\![\frac{q_A(x)}{}]\!] + [\![\frac{q_B(x)}{q_{A \parr B}(x)}]\!]$.

8 — CORRECTNESS

- ***Gabarit:*** **all ordeals obtained by *switching* the $\parr$-links.**

  Vehicles coloured in `blue`.

  Correctness: $\mathcal{V} + \mathcal{O}$ strongly normalises into

  Normal form: $[\![\, p_\Gamma(x) \,]\!] := [\![\, \{p_A(x); A \in \Gamma\} \,]\!]$.

- **η-expansion: identity links on literals. Criterion insensitive.**

- ***Herbrand:*** **existentials as functions of universals $\vec{y} = \vec{t}[\vec{x}]$.**

  $x := f(y)$ as independence of $y = t$ from x, i.e., $\exists y \forall x$.

- **X ($\sim X$) must be paired; not with $X, Y, \sim Y$ ($\sim X, Y, \sim Y$).**

  Essentialism: complementarity of *names*.

  Literal $X, \sim X$: occ. of *universally* quantified variable $\forall X$.

  Cancelling ordeal: special kind bound to normalise to $\emptyset$.

  Switching: select a literal in all pairs, $\sim X, \sim Y, Z$.

  Sum of all: $[\![\, \frac{q_A(\mathtt{l}\cdot x), q_A(\mathtt{r}\cdot x)}{} \,]\!]$ when literal A is selected.

9 — THE CUT RULE

- Cut as conclusion $[A \otimes \sim A]$. *Predicts* erasure, usage.
- *Vehicle* $\mathcal{V}$ with conclusions $\vdash \Gamma, A \otimes \sim A$ and
  Feedback: $\mathcal{F}_A := [\![\frac{p_A(x), p_{\sim A}(x)}{}]\!]$; fits $p_A(-)$ and $p_{\sim A}(-)$.
  Performance: $\mathcal{V} + \mathcal{F}_A$ possibly yields normal form $\mathcal{W}$.
  Correctness: of $\mathcal{W}$ w.r.t. *ordeal* $\mathcal{O}$ for $\vdash \Gamma$.
- Case $A = X$: $\mathcal{V} = [\![\frac{}{p_{\sim X'}(x), p_X(x)}]\!] + [\![\frac{}{p_{\sim X}(x), p_{X"}(x)}]\!] + \ldots$
  Then: $\mathcal{W} = [\![\frac{}{p_{\sim X'}(x), p_{X"}(x)}]\!] + \ldots$ passes test $\mathcal{O}$.
- Case $A = B \otimes C$; replace $\mathcal{F}_A$ with $\mathcal{F}_B + \mathcal{F}_C$.
  Change of syntheticity: two cuts instead of one.
  $\mathcal{V} + \mathcal{F}_A$ same normal form as $\mathcal{V} + \mathcal{F}_B + \mathcal{F}_C$.
- Replacing $[\![\frac{q_D(x)}{p_D(x)}]\!]$ with $[\![\frac{q_D(x)}{}]\!]$ in $\mathcal{O}$ yields *closing* $\mathcal{O}'$.
  Main result: $\mathcal{V} + \mathcal{O}'$ normalises into:
  $[\![\frac{}{p_B(x)}]\!] + [\![\frac{}{p_C(x)}]\!] + [\![\frac{}{p_{\sim B}(x), p_{\sim A}(x)}]\!]$

10 — EXPONENTIALS REVISITED

- *Knitting:*, e.g., *Church-Rosser* with two pairs of colours.
  Relates usine/usage: compositionality, BHK.
- Poorly knitted operations only live at *second order*.
  Exponentials: $!A, ?A$.
  Intuitionistic disjunction: $!A \oplus !B$; *commutative* cuts.
  Multiplicative neutrals: $1, \bot$.
- Basic problem: *weakening* impossible.
  Want of *physical* connection.
  Hidden conclusion: $\Gamma, \underline{\Delta}$.
  Ordeal: $[\![\frac{q_A(x)}{}]\!]$ when $A \in \Delta$ hidden.
- Revert to *intuitionistic* implication... Not quite.
  Semi-tensor $A \oslash B := !A \otimes B$.
  Semi-par $A \ltimes B := ?A \parr B$.
- Vehicles: auxiliary variable for *duplication:* $p_A(x \cdot y)$.

III — WHAT CONVEYS CERTAINTY?

Keywords: derealism, epidictic, épure.

11 — DEREALISM

- **First order treatment of $\mathbb{N}$ *axiomatic*, $\neq$ logic.**

  Second order: (Dedekind) induction on T handled by $\exists X$.
  Flexibility: range of (inductive) witnesses T in $A[T/X]$.
  Subf. property: depends on possible T; ditto for 1st order.
  Foundational problems: reduction usage/usine problematic.

- ***Church* and *Curry* both wrong w.r.t. l'usine:**

  Essentialism: objets born synthetic, *typed*. No usine.
  Existentialism: objects born analytic, *untyped*. Usine ∞.

- ***Derealism:* usine stays finite if witness made part of proof.**

  Épure: analytic *vehicle* + synthetic *mould*, i.e., witness.
  Epidictics: requires/believes moulds to be *balanced*.
  Balance: rights/duties (cut-elim.) not checkable at usine.

- ***Consistency* and Hegel's contradictory foundations:**
  Animæ: « Incorrect » proofs, mingle analytic/synthetic.

12 — SYSTEM F

- *Second order* quantifications: over *propositions*.

  Links:

$$\frac{A}{\forall X\, A} \qquad\qquad \frac{A[T/X]}{\exists X\, A}$$

- Can be handled by *usine* (proof-nets).

  $\forall X$: $X := \cdot / \otimes / \parr$, hence $\sim X := \cdot / \parr / \otimes$.

  Existential $\exists X$: T provides its own switchings.

- However, T is part of the *derealist* answer.

  Épure: combination vehicle + *mould*, e.g., $T + \sim T$.

  Balance: how do we know that $T + \sim T$ actually match ?

  Object/Subject no longer valid: answer partly *subjective*.

  Answer combines analytic and synthetic features.

  Epidictic: uncheckable affirmation.

13 — ANIMÆ

- Derealism: two pairs, `blue`/`yellow` and `red`/`cyan`.

  Animæ: uses colours `blue`, `red`.

  Épure: splits as $\mathcal{V} + \mathcal{M}$.

  Animist otherwise: Object and Subject intertwinned.

  Ordeal: uses colours `yellow`, `cyan`, `black`.

- Additive neutrals: no balance problem in $\exists X X$.

  $\top$: unique ordeal $[\![\frac{R(x), S(x)}{}]\!] + [\![\frac{T(x)}{\top(x)}]\!]$.

  0: three ordeals, $[\![\frac{r(x)}{}]\!] + [\![\frac{s(x), t(x)}{O(x)}]\!]$ and

  $[\![\frac{s(x)}{}]\!] + [\![\frac{r(x), t(x)}{O(x)}]\!]$ and $[\![\frac{t(x)}{O(x)}]\!]$.

- The absurdity has an *animist* proof:

  $[\![\frac{}{t(x)}]\!] + [\![\frac{}{r(x), s(x)}]\!]$.

  But no épure: hence consistency.